Gravitation und Kosmologie

III: Das inflationäre Universum

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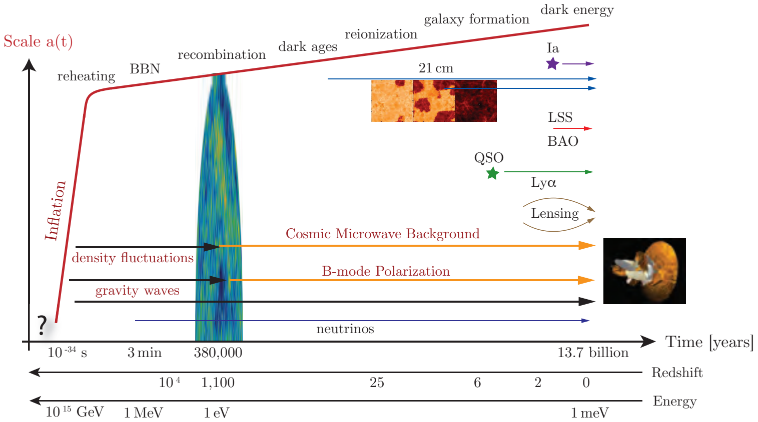
Inhalt von Teil III

Geschichte des Universums



Abbildungsnachweis: D. Baumann, arXiv:0907.5424v2

The CMB spectrum from the PLANCK mission

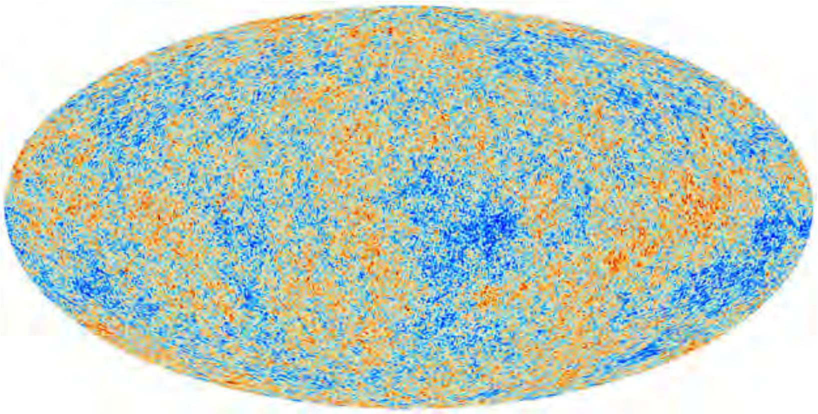


Figure credit: ESA/PLANCK Collaboration

$$T_0 = (2.7260 \pm 0.0013)\mathrm{K}$$

Particle horizon

Defining conformal time η, the metric for a Friedmann–Lemaître universe can be written as

$$\mathrm{d}s^2 = a^2(\eta)\left[-\mathrm{d}\eta^2 + (\mathrm{d}\chi^2 + f_k^2(\chi)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]$$

Radial light propagation:

$$\mathrm{d}s^2 = 0 = a^2(\eta)\left(-\mathrm{d}\eta^2 + \mathrm{d}\chi^2\right)$$

Maximal comoving distance light can travel from t_i to t:

$$\chi_\mathrm{p}(\eta) = \eta - \eta_\mathrm{i} = \int_{t_\mathrm{i}}^{t} \frac{\mathrm{d}t}{a(t)}$$

Particle horizon

The expression for the particle horizon can be written as

$$\eta - \eta_\mathrm{i} = \int_{t_\mathrm{i}}^{t} \frac{\mathrm{d}t}{a(t)} = \int_{a_\mathrm{i}}^{a} \mathrm{d}\ln a \, \frac{1}{aH},$$

with $(aH)^{-1}$ as the comoving Hubble radius.

During the radiation and matter dominated phases, $(aH)^{-1}$ grows monotonically. This means that the fraction of the universe in causal contact increases with time. In the past, therefore, a smaller fraction of the universe had causal contact.

Horizon problem

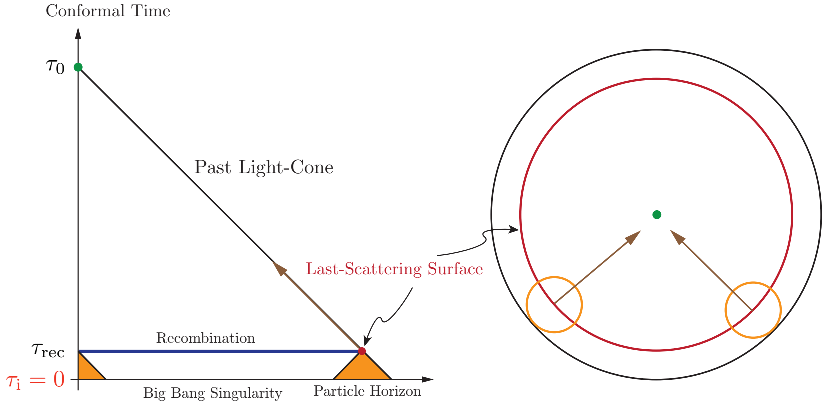


Figure credit: D. Baumann, arXiv:0907.5424v2

Inflation

Demand a decrease of the comoving Hubble radius with time,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{aH}\right) < 0,$$

which leads to

$$\ddot{a} > 0 \Rightarrow \rho + 3p < 0.$$

For $\rho \approx -p \approx \mathrm{constant}$, one gets a (quasi-)exponential expansion of the universe. This is called inflation.

We have

$$|1 - \Omega| = \frac{1}{(aH)^2}.$$

Inflation predicts that the universe evolves towards flatness,

$$\Omega = 1 \pm 10^{-5}.$$

Observation gives $\Omega = 1 \pm 0.02$.

Solution of the horizon problem

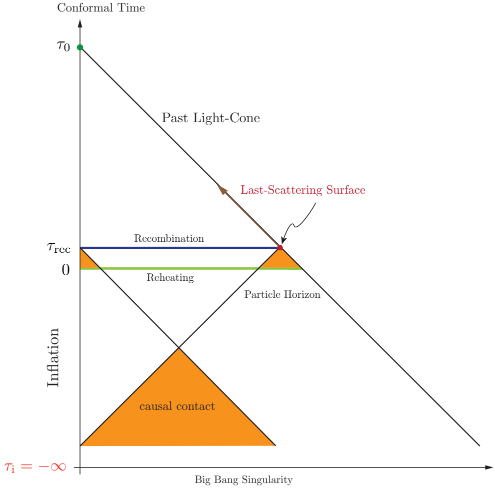


Figure credit: D. Baumann, arXiv:0907.5424v2

Dynamics of inflation

Assume the occurrence of a quasi-exponential expansion phase around 10^{-34} seconds after the Big Bang. Many models involve a scalar field ϕ ('inflaton') and a phase of slow-roll.

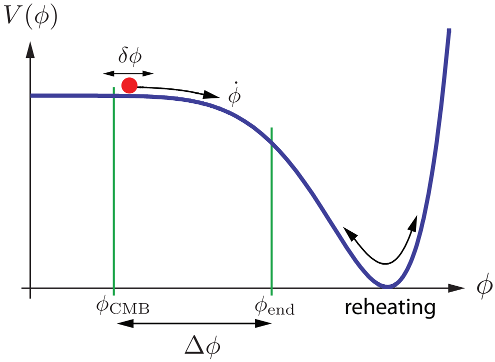


Figure credit: D. Baumann, arXiv:0907.5424v2

Slow-roll inflation

Demand the smallness of the slow-roll parameters,

$$\epsilon_V := \frac{M_{\mathrm{P}}}{2}\left(\frac{V_{,\phi}}{V}\right)^2 \ll 1, \quad \eta_V := M_{\mathrm{P}}^2 \frac{V_{,\phi\phi}}{V} \ll 1,$$

where $M_{\mathrm{P}} = \sqrt{\hbar/8\pi G}$ is the (reduced) Planck mass.

Number of e-folds before the end of inflation:

$$N(\phi) := \ln \frac{a_{\mathrm{end}}}{a} = \int_t^{t_{\mathrm{end}}} H \mathrm{d}t \approx \int_{\phi_{\mathrm{end}}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{2\epsilon_V}}$$

One requires $N > 60$ to solve the horizon and flatness problems.

Cosmic perturbation theory

Perturbations around the homogeneous background:

$$\phi(t,\mathbf{x}) = \bar{\phi}(t) + \delta\phi(t,\mathbf{x}), \qquad g_{\mu\nu}(t,\mathbf{x}) = \bar{g}_{\mu\nu}(t) + \delta g_{\mu\nu}(t,\mathbf{x}),$$

with

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \\ &= -(1+2\Phi)\mathrm{d}t^2 + 2aB_k \mathrm{d}x^k \mathrm{d}t + a^2\left[(1-2\Psi)\delta_{kl} + E_{kl}\right]\mathrm{d}x^k\mathrm{d}x^l \end{aligned}$$

We further decompose

$$B_k := \partial_k B - S_k, \qquad \partial^k S_k = 0,$$

$$E_{kl} := 2E_{,kl} + 2\partial_{(k}F_{l)} + h_{kl}, \quad \partial^l F_l = 0, \;\; h^k_{\;k} = 0 = \partial^k h_{kl}$$

The h_{kl} are the (gauge-invariant) tensor modes, which describe gravitational waves (classical theory) resp. gravitons (quantum theory).

Appropriate variables

- Gauge-invariant comoving curvature perturbation:

$$\mathcal{R} := \Psi + \frac{H}{\dot{\phi}}\delta\phi$$

- Mukhanov–Sasaki variable:

$$v := z\mathcal{R}, \quad z^2 := a^2 \frac{\dot{\phi}^2}{H^2}$$

  Fourier space: $v_{\mathbf{k}}$

- Tensor modes: decompose h_{kl} into Fourier components $h^s_{\mathbf{k}}$ (for two polarizations $s = 1, 2$) and define

$$v^s_{\mathbf{k}} := \frac{aM_{\mathrm{P}}}{2} h^s_{\mathbf{k}}$$

The action for the $v^s_{\mathbf{k}}$ consists essentially of two copies of the action for the $v_{\mathbf{k}}$.

Quantum origin of the perturbations

In the inflationary scenario, the perturbations $v_{\mathbf{k}}$ and $v^s_{\mathbf{k}}$ are quantum degrees of freedom. It is generally assumed that they are in their ground state at the onset of inflation (for wave numbers $k \ll aH$).

- Heisenberg picture:

$$\hat{v} = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \left[v_k(\tau) \hat{a}_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{kx}} + v_k^*(\tau) \hat{a}^\dagger_{\mathbf{k}} \mathrm{e}^{-\mathrm{i}\mathbf{kx}} \right]$$

- Schrödinger picture:

$$\psi_k(v_k, \tau) = N_k \exp\left(-\Omega_k(\tau)|v_k|^2\right)$$

Important feature of inflation: For wave numbers $k < aH$, the quantum state evolves into an excited state (in the Schrödinger picture, a two-modes squeezed state)

Next step: Calculation of two-point functions (power spectrum)

Scalar perturbations

For $k \ll aH$, the perturbations develop classical properties by decoherence[1] and are characterized by

$$\langle \mathcal{R}_{\mathbf{k}} \mathcal{R}_{\mathbf{k}'} \rangle =: (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') P_{\mathcal{R}}(k), \quad \Delta_{\rm s}^2(k) := \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k)$$

Spectral index: $n_{\rm s} - 1 := \mathrm{d} \ln P_{\mathcal{R}} / \mathrm{d} \ln k$

Explicit calculations give

$$\Delta_{\rm s}^2(k) = \frac{1}{8\pi^2} \left(T_{\rm P} H\right)^2 \epsilon_V^{-1} \approx 2 \times 10^{-9}$$

The amplitude on the right-hand side is fixed by observation. Since this expression contains the Planck time, it is a quantum gravitational effect!

(H is evaluated at Hubble-scale exit, that is, at $k = aH$.)

Mukhanov and Chibisov (1981); Hawking (1982); Guth and Pi (1982); Starobinsky (1982)

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[1] C.K., Polarski, and Starobinsky (1998), . . .

The CMB spectrum from the PLANCK mission

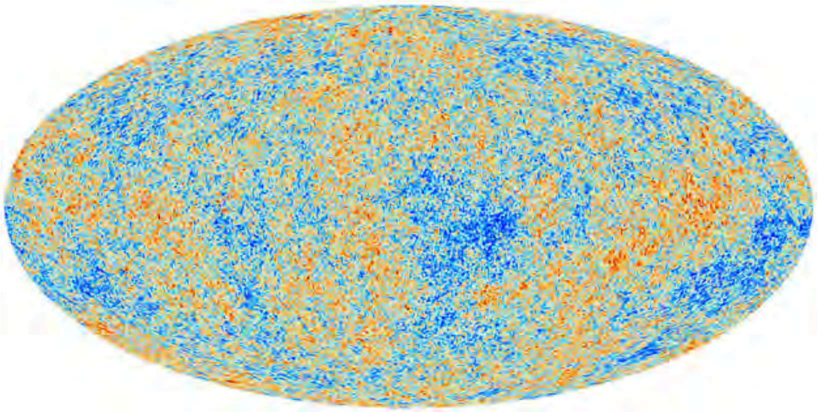


Figure credit: ESA/PLANCK Collaboration

Temperature anisotropies

Expand temperature anisotropies in spherical harmonics,

$$\frac{\Delta T(\hat{n})}{T_0} = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n})$$

For the observations:

$$C_\ell := \frac{1}{2\ell + 1} \sum_m \langle a^*_{\ell m} a_{\ell m} \rangle$$

C_ℓ can be connected to the power spectrum $\Delta^2_{\rm s}(k)$.

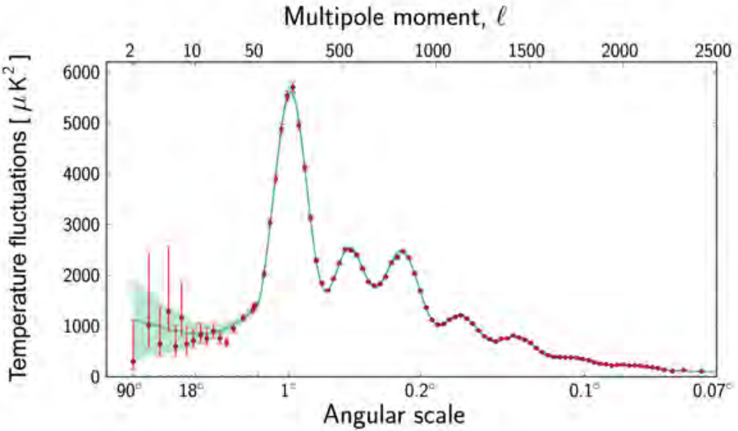


Figure credit: ESA/PLANCK Collaboration

Tensor perturbations

The quantized tensor perurbations describe the gravitons produced during inflation

$$\langle h_{\mathbf{k}} h_{\mathbf{k}'} \rangle =: (2\pi)^3 \delta(\mathbf{k}+\mathbf{k}') P_{\mathcal{R}}(k), \quad \Delta^2_{\mathrm{h}}(k) := \frac{k^3}{2\pi^2} P_{\mathcal{R}}(k)$$

Explicit calculations give

$$\Delta^2_{\mathrm{t}}(k) := 2\Delta^2_{\mathrm{h}}(k) = \frac{2}{\pi^2} \left(T_{\mathrm{P}} H\right)^2$$

(H is evaluated at Hubble-scale exit, that is, at $k = aH$.)
The discovery of primordial tensor perturbations would be a genuine quantum gravitational effect.

Starobinsky (1979)

r-parameter

Tensor-to-scalar ratio: $r := \frac{\Delta_{\mathrm{t}}^2}{\Delta_{\mathrm{s}}^2} = 16\epsilon$

Knowing r, one knows the energy scale of inflation,

$$\mathcal{E}_{\mathrm{inf}} \approx 1.06 \times 10^{16}\ \mathrm{GeV} \left(\frac{r}{0.01}\right)^{1/4}$$

Large r-values thus correspond to inflation at the GUT-scale.

Information about r is contained in the polarization of the CMB.

The BICEP2 experiment

"Background Imaging of Cosmic Extragalactic Polarization"

Figure credit: BICEP2 Collaboration

In 2014, the BICEP2 collaboration announced the result

$$r = 0.20^{+0.07}_{-0.05}$$

- In a joint analysis of BICEP2, PLANCK, and KECK (2015), it was concluded that $r < 0.12$ at 95% confidence limit.
- PLANCK (2015): $r < 0.09$

Starobinsky inflation

The model proposed by Starobinsky in 1980 – the first model of inflation – is currently the most successful one with regard to observations.

- In this model, a curvature-squared term is added to the action,

$$R \longrightarrow R + c_2 R^2;$$

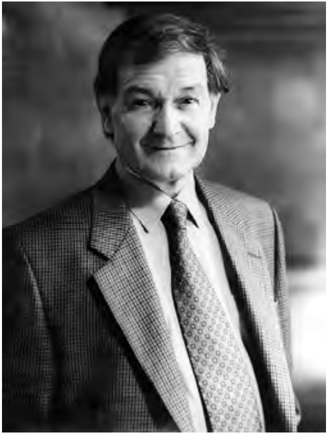
  the acceleration comes from the R^2-term.

- This action is classically equivalent to an action with the Einstein–Hilbert term and a scalar field ("Einstein frame"); the scalar field has the potential

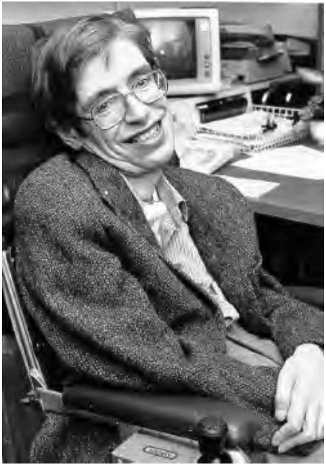
$$V(\phi) = \frac{1}{8c_2}\left(1 - \exp\left(-\sqrt{\frac{2}{3}}\sqrt{8\pi G}\,\phi\right)\right)^2$$

- The model predicts $n_\mathrm{s} \approx 1 - \frac{2}{N} \in (0.960, 0.967)$ and $r \approx \frac{12}{N^2} \in (0.003, 0.005)$; compare this with the observed values (PLANCK 2015): $n_\mathrm{s} = 0.9667 \pm 0.0040$ and $r < 0.09$.

Die Grenzen der klassischen Kosmologie



(a) Roger Penrose



(b) Stephen Hawking

Roger Penrose und Stephen Hawking 1970:

Unter sehr allgemeinen Annahmen ist Einsteins Allgemeine Relativitätstheorie unvollständig; sie kann den Anfang des Universums nicht beschreiben („Singularitätentheoreme").

QUANTENGRAVITATION?

Literatur

- G. Calcagni, *Classical and Quantum Cosmology* (Springer)
- V. Mukkanov, *Physical Foundations of Cosmology* (Cambridge University Press)
- P. Peter und J.-P. Uzan, *Primordial Cosmology* (Oxford University Press)