## *Cosmology and the origin of structure*

**Rocky I:** The universe observed

**Rocky II:** Perturbations

**Rocky III:** Inflation

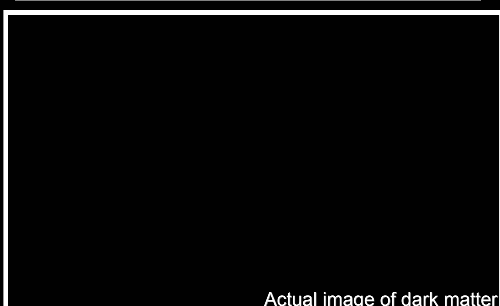
http://home.fnal.gov/~rocky/maria_laach_1.pdf
http://home.fnal.gov/~rocky/maria_laach_2.pdf
http://home.fnal.gov/~rocky/maria_laach_3.pdf

**Herbstschule für Hochenergiephysik Maria Laach**
***Rocky Kolb***
***Fermilab & The University of Chicago***

## Observer's view of the universe


**lumpy (inhomogeneous and anisotropic)**
**full of stars, galaxies, clusters, ….**

## Theorist's view of the universe




**smooth (homogeneous and isotropic)**
**full of dark matter (and dark energy)**

## *Power spectrum*

- Assume there is an average density $\bar{\rho}$
- Expand density contrast $\delta(\vec{x})$ in Fourier modes

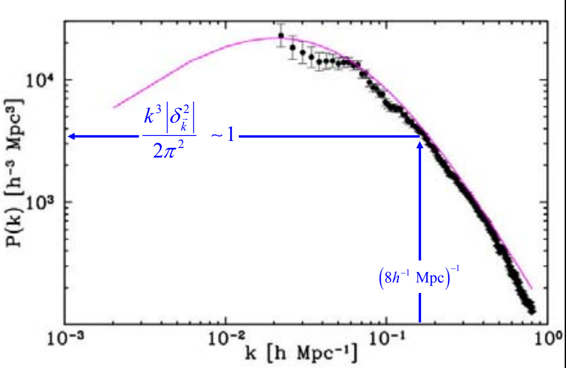
$$\delta(\vec{x}) \equiv \frac{\rho(\vec{x}) - \bar{\rho}}{\bar{\rho}} = \int \delta_{\vec{k}} \exp\left(-i\vec{k}\cdot\vec{x}\right) d^3k$$

- Autocorrelation function defines power spectrum

$$\left\langle \frac{\delta\rho(\vec{x})}{\bar{\rho}} \right\rangle^2 = \langle \delta(\vec{x})\delta(\vec{x}) \rangle = \int_0^\infty \frac{dk}{k} \frac{k^3 |\delta_{\vec{k}}^2|}{2\pi^2}$$

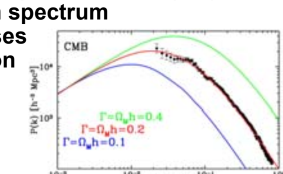
$$\Delta^2(k) \equiv \frac{k^3 |\delta_{\vec{k}}^2|}{2\pi^2} \qquad P(k) \equiv |\delta_{\vec{k}}^2|$$

## *Power spectrum*




## *Rocky II: Growth of structure*

- **Linear regime: quantative analysis**
  **Jeans analysis**
  **Sub-Hubble-radius perturbations (Newtonian)**
  **Super-Hubble-radius perturbations (GR)**
  **Harrison-Zel'dovich spectrum**
  **Dissipative processes**
  **The transfer function**
  **Linear evolution**




- **Non-linear regime: word calculus**
  **Comparison to observations**
  **A few clouds on the horizon**

## Growth of small perturbations

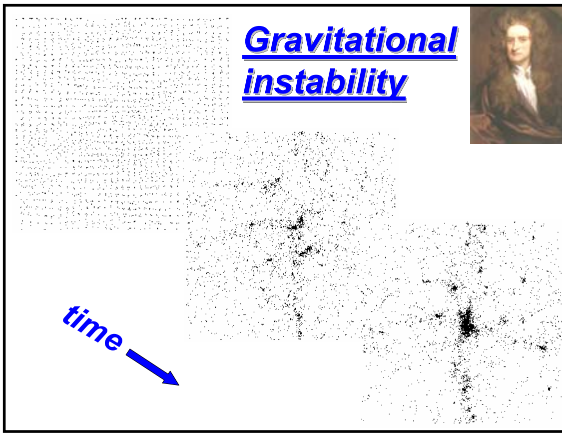
**Today (14 Gyr AB)**

- radiation and matter decoupled
- $\Delta T / T \sim 10^{-5}$
- $\Delta \rho_G / \rho_G \sim 10^{+6}$

**Before recombination (300 kyr AB)**

- radiation and matter coupled
- $\Delta T / T \sim 10^{-5}$
- $\Delta \rho_G / \rho_G \sim 10^{-5}$

## Gravitational instability




## Everything in the universe:

**Clusters**
**Galaxies**
**Stars**
**Planets**
**Poodles**
**Pigeons**
**Pond Scum**
**Donald Rumsfeld**
**.**
**.**

**From inflationary perturbations**

## Jeans analysis

Jeans analysis in a non-expanding fluid:

| | | |
|---|---|---|
| matter density | $\rho$ | |
| pressure | $p$ | |
| velocity field | $\vec{v}$ | |
| gravitational potential | $\phi$ | |

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0 \\ \dfrac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} + \dfrac{1}{\rho} \vec{\nabla} p + \vec{\nabla} \phi = 0 \\ \nabla^2 \phi = 4\pi G \rho \end{cases}$$

Perturb about solution*

| | |
|---|---|
| $\rho = \rho_0 = \text{constant}$ | $\rho = \rho_0 + \rho_1$ |
| $p = p_0 = \text{constant}$ | $p = p_0 + p_1$ |
| $\vec{v} = 0 = \text{constant}$ | $\vec{v} = \vec{v}_0 + \vec{v}_1$ |
| $\phi = \phi_0 = \text{constant}$ | $\phi = \phi_0 + \phi_1$ |

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla^2 \rho_1 = 4\pi G \rho_1$$

$$v_s^2 = p_1 / \rho_1$$

## Jeans analysis

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla^2 \rho_1 = 4\pi G \rho_1$$

solutions of the form $\rho_1(\vec{r}, t) = \delta(\vec{r}, t) \rho_0 = A_k \exp\left(-i\vec{k} \cdot \vec{r} + i\omega t\right)$

$\omega$ and $k$ satisfy the dispersion relation $\omega^2 = v_s^2 k^2 - 4\pi G \rho_0$

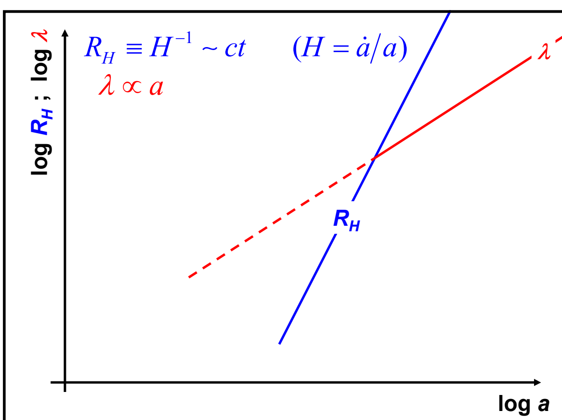
$\omega$ real: perturbations oscillate as sound waves

$\omega$ imaginary: exponentially growing (or decaying) modes

Jeans wavenumber $k_J = \left( \dfrac{4\pi G \rho_0}{v_s^2} \right)^{1/2}$ — $k > k_J$ perturbation oscillates; $k < k_J$ perturbation grows

Jeans mass $M_J = \dfrac{4\pi}{3} \left( \dfrac{\pi}{k_J} \right)^3 \rho_0$ — $M < M_J$ perturbation oscillates; $M > M_J$ perturbation grows

gravitational pressure vs. thermal pressure



## Sub-Hubble-radius ($R_H$=$H^{-1}$)

Jeans analysis in an expanding fluid:
scale factor *a(t)* describes expansion,
unperturbed solution:

$$\rho_0 = \rho_0(t_0)\, a^{-3}(t) \qquad \vec{v}_0 = \frac{\dot a}{a}\vec r \qquad \vec\nabla\phi_0 = \frac{4\pi G\rho_0}{3}\vec r$$

$$\ddot\delta_k + 2\frac{\dot a}{a}\dot\delta_k + \left(\frac{v_s^2k^2}{a^2} - 4\pi G\rho_0\right)\delta_k = 0$$

- Solution is some sort of Bessel function: growth or oscillation depends on Jeans criterion
- In matter-dominated era $\rho_0 = \left(6\pi G t^2\right)^{-1}$ and $\dot a/a = 2/3t$
- For wavenumbers less than Jeans

$$\delta_+(t) = \delta_+(t_i)\left(t/t_i\right)^{2/3} \qquad \delta_-(t) = \delta_-(t_i)\left(t/t_i\right)^{-1}$$

## Super-Hubble-radius ($R_H$=$H^{-1}$)

$$g_{\mu\upsilon}(\vec x,t) = g_{\mu\upsilon}^{FRW}(t) + \delta g_{\mu\upsilon}(\vec x,t)$$
$$T_{\mu\upsilon}(\vec x,t) = T_{\mu\upsilon}^{FRW}(t) + \delta T_{\mu\upsilon}(\vec x,t)$$
$$\delta R_{\mu\upsilon} - (1/2)\delta\left[g_{\mu\upsilon}R\right] = 8\pi G\delta T_{\mu\upsilon}$$

- complete analysis not for the faint of heart
- interested in "scalar" perturbations
- fourth-order differential equation
- only two solutions "physical"
- other two solutions are "gauge modes" which can be removed by a coordinate transformation on the unperturbed metric

$$\delta R_{\mu\upsilon} - (1/2)\delta\left[g_{\mu\upsilon}R\right] = 8\pi G\delta T_{\mu\upsilon}$$

Bardeen 1980

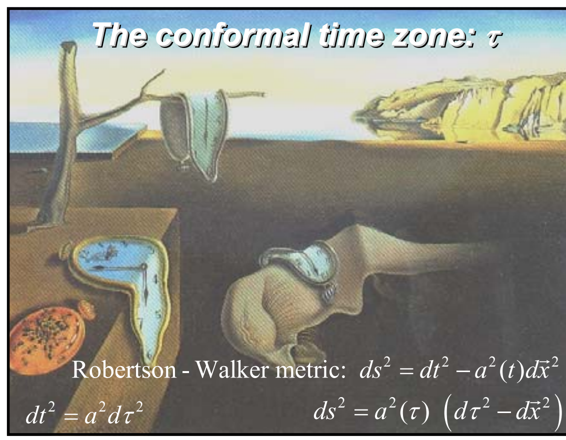
**Reference spacetime: flat FRW**

$$ds^2 = a^2(\tau)\left\{d\tau^2 - \delta_{ij}dx^i dx^j\right\}$$

$$\tau = \text{ conformal time}$$

$$dt^2 = a^2(\tau)d\tau^2$$

## The conformal time zone: $\tau$



Robertson - Walker metric: $ds^2 = dt^2 - a^2(t)d\vec x^2$

$dt^2 = a^2 d\tau^2$ $\qquad ds^2 = a^2(\tau)\ \left(d\tau^2 - d\vec x^2\right)$


$$\delta R_{\mu\upsilon} - (1/2)\delta\left[g_{\mu\upsilon}R\right] = 8\pi G\delta T_{\mu\upsilon}$$

Bardeen 1980

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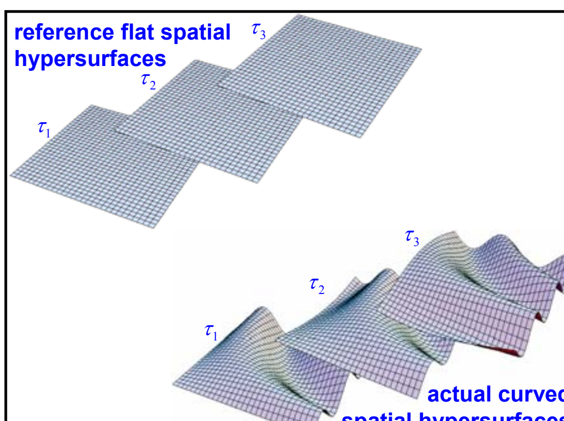
$$\tau = \text{ conformal time}$$

$$dt^2 = a^2(\tau)d\tau^2$$


**Perturbed spacetime (10 degrees of freedom):**

$$ds^2 = a^2(\tau)\Big\{\left(1+\delta g_{00}\right)d\tau^2$$
$$-2\delta g_{0i}d\tau\ dx^i - \left(\delta_{ij} + 2\delta g_{ij}\right)dx^i dx^j\Big\}$$



## scalar, vector, tensor decomposition

$\delta g_{00} = 2A$ — *1*

$\delta g_{0i} = S_i + \partial_i B$ — *2+1*

$(\partial^i S_i = 0)$

$\delta g_{ij} = h_{ij} - \psi\delta_{ij} + \partial_i F_j + \partial_j F_i + \partial_i \partial_j E$ — *2+1+2+1*

$(h^i{}_i = 0 \; ; \; \partial^i h_{ij} = 0 \; ; \; \partial^i F_i = 0)$ — *10*

**evolution of scalar, vector, and tensor perturbations decoupled**

## Vector Perturbations:

- are not sourced by stress tensor
- decay rapidly in expansion

## Tensor Perturbations:

- perturbations of transverse, traceless component of the metric: gravitational waves
- do not couple to stress tensor

## Scalar Perturbations

- couple to stress tensor
- *density perturbations!*

## Scalar perturbations

$\delta g_{00} = \psi$ **scalar perturbations** **(4 d.o.f.)**

$2\delta g_{ij} = -\psi\delta_{ij}$

**Gauge freedom:** **choose gauge** ***B=E=0*** **(2 d.o.f.)**
(longitudinal, conformal-Newtonian)

**Stress tensor:** **isotropic stress** ***A=ψ*** **(1 d.o.f.)**

**Scalar metric perturbations:**

$$ds^2 = a^2(\tau)\left\{\left(1+\psi\right)d\tau^2 - \left(1-\psi\right)d\vec{x}^2\right\}$$

## Gauge invariant variable $\Re$

**Scalar metric perturbations:**

$$ds^2 = a^2(\tau)\left\{\left(1+\psi\right)d\tau^2 - \left(1-\psi\right)d\vec{x}^2\right\}$$

***Gauge invariant variable:*** $\Re$

***intrinsic curvature perturbations on comoving hypersurfaces***

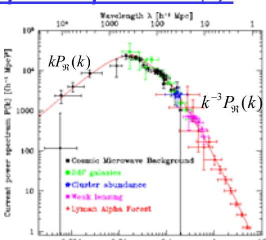
## Gauge invariant variable $\Re$

***Expand in Fourier modes:***

$$\Re = \int \frac{d^3k}{(2\pi)^{3/2}} \Re_k \, e^{ik\cdot x} \qquad \frac{k^3}{2\pi^2}\langle \Re_k \Re_k \rangle = P_\Re(k)$$

***Relate to the observed power spectrum P(k):***

$$\frac{k^3}{2\pi^2}P(k) = \left(\frac{k}{aH}\right)^4 T^2(k) P_\Re(k)$$




***Transfer function:***

$$T^2(k) \rightarrow \begin{cases} 1 & k/aH \rightarrow 0 \\ k^{-4} & k/aH \rightarrow \infty \end{cases}$$

## Super-Hubble-radius

in synchronous gauge $A = B = 0$

and uniform Hubble flow gauge $B = E = 0$

$\delta_+(t) = \delta_+(t_i)\left(t/t_i\right)^{2/3}$ matter-dominated

$\delta_+(t) = \delta_+(t_i)\left(t/t_i\right)$ radiation-dominated

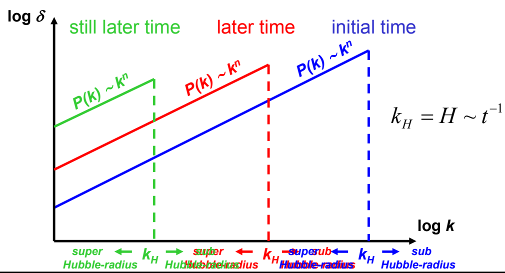
- in matter-dominated era
  - $\delta_+(t) = \delta_+(t_i)\left(t/t_i\right)^{2/3}$ scales larger than Hubble radius
  - $\delta_+(t) = \delta_+(t_i)\left(t/t_i\right)^{2/3}$ scales smaller than Hubble radius
- in radiation-dominated era
  - $\delta_+(t) = \delta_+(t_i)\left(t/t_i\right)$ scales larger than Hubble radius
  - $\delta_+(t) = \text{constant}$ scales smaller than Hubble radius

# Harrison-Zel'dovich

in radiation-dominated era

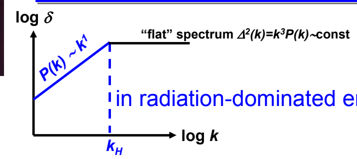
$\delta_+(t) = \delta_+(t_i)(t/t_i)$ scales larger than Hubble radius

$\delta_+(t) = \text{constant}$ scales smaller than Hubble radius



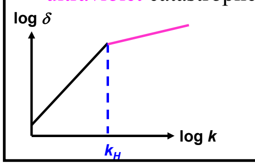

$k_H = H \sim t^{-1}$

# Harrison-Zel'dovich



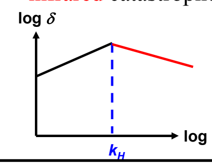

$P(k) \propto k^n \quad n > 1$

ultraviolet catastrophe



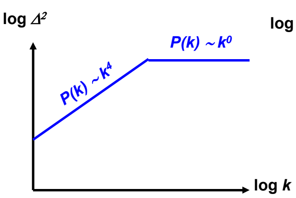

$P(k) \propto k^n \quad n < 1$

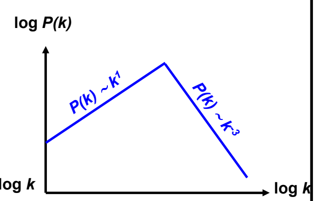
infrared catastrophe




# The power spectrum

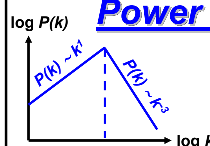
$$\Delta^2(k) \equiv \frac{k^3 |\delta_{\vec{k}}^2|}{2\pi^2} \qquad P(k) \equiv |\delta_{\vec{k}}^2| \sim k^{-3}\Delta^2(k)$$



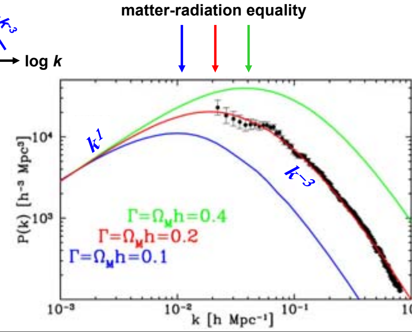




- in radiation-dominated era no growth sub-Hubble radius growth as *t* super-Hubble radius
- in matter-dominated era power spectrum grows as $t^{2/3}$ on all scales

# Power spectrum for CDM




matter-radiation equality



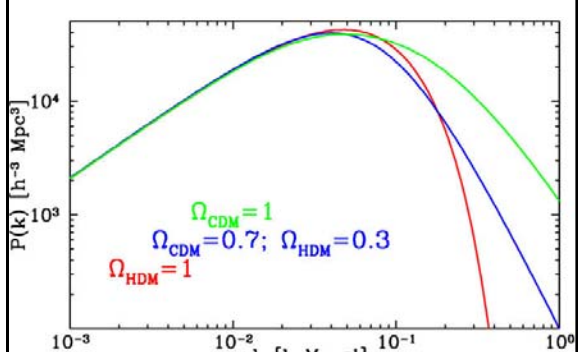

# Dissipative processes

1. **Collisionless phase mixing – free streaming**

   If dark matter is relativistic or semi-relativistic particles can stream out of overdense regions and smooth out inhomogeneities. The faster the particle the longer its free-streaming length.

   Quintessential example: eV-range neutrinos

# The evolved spectrum



# Dissipative processes

1. **Collisionless phase mixing – free streaming**

   If dark matter is relativistic or semi-relativistic particles can stream out of overdense regions and smooth out inhomogeneities. The faster the particle the longer its free-streaming length.

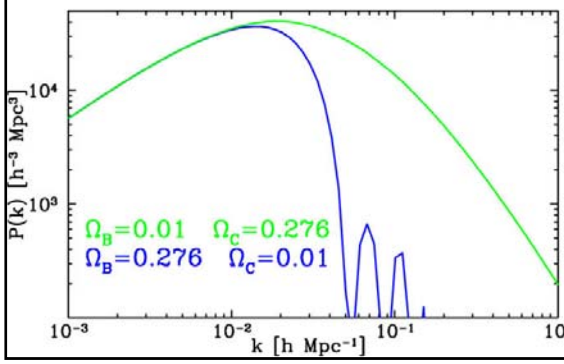
   Quintessential example: eV-range neutrinos

2. **Collisional damping – Silk damping**

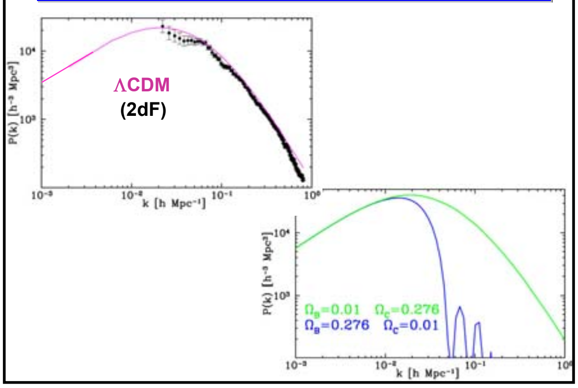
   As baryons decouple from photons, the photon mean-free path becomes large. As photons escape from dense regions, they can drag baryons along, erasing baryon perturbations on small scales.

   Baryon-photon fluid suffers damped oscillations.

# The evolved spectrum

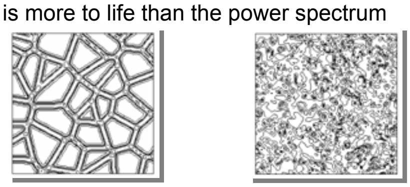



# Silk damping of baryon pert'ns

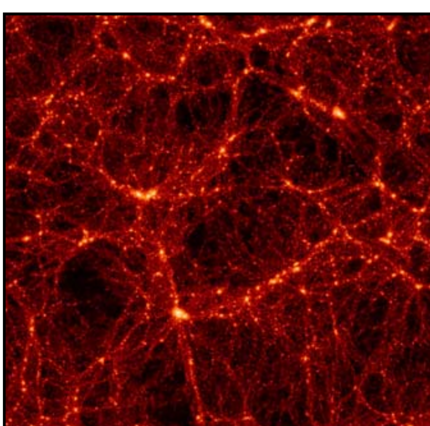



# Life ain't linear!

- Many scales become nonlinear at about the same time
- Mergers from many smaller objects while larger scales form
- N-body simulations for dissipation-less dark matter
- Hydro needed for baryons
- Power spectrum well fit if $\Gamma = \Omega h \sim 0.2$
- There is more to life than the power spectrum

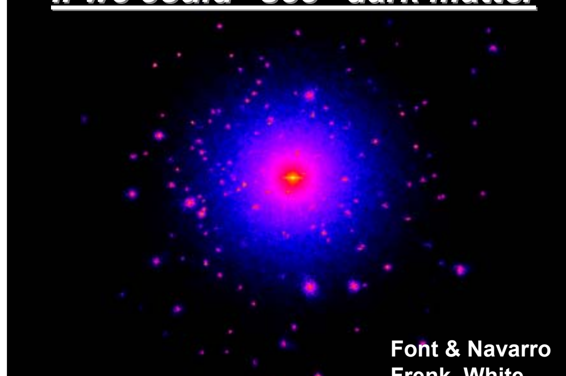


Alex Szalay

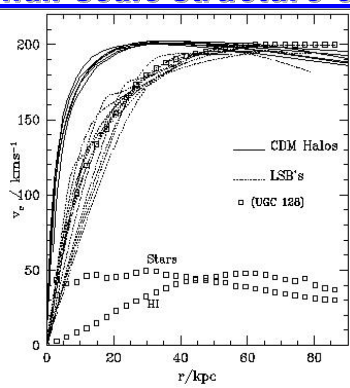


**Large-scale structure fits well**

# If we could "see" dark matter

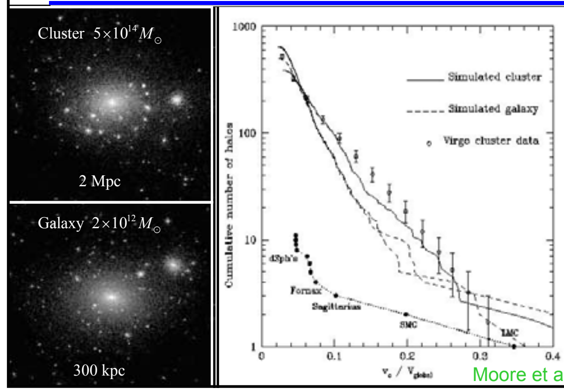


## Small-scale structure-cusps




Moore et al.

## Small-scale structure-satellites




Moore et al.

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**Herbstschule für Hochenergiephysik Maria Laach**
***Rocky Kolb***
***Fermilab & The University of Chicago***