Renormalization

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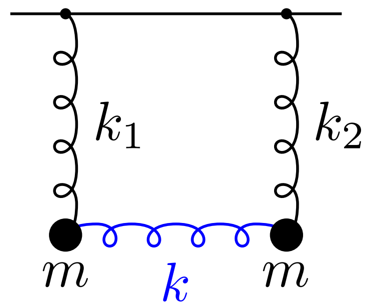
- Physical meaning of renormalization
- Ultraviolet divergences and regularization
- Renormalizable and non-renormalizable theories
- Renormalization of the ϕ^4 theory
- Renormalization of gauge theories and the Electroweak Standard Model

Coupled oscillators

$$L = \frac{m}{2}(\dot{x}_1^2 - \omega_1^2 x_1^2) + \frac{m}{2}(\dot{x}_2^2 - \omega_2^2 x_2^2) - \frac{k}{2}(x_1 - x_2)^2$$

$$\omega_{1,2}^2 = \frac{k_{1,2}}{m}$$



$$\kappa^2 = \frac{k}{m}$$

$$\mathbf{x}_0(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (a_1 \mathrm{e}^{\mathrm{i}\omega_1 t} + a_2 \mathrm{e}^{-\mathrm{i}\omega_1 t}) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (b_1 \mathrm{e}^{\mathrm{i}\omega_2 t} + b_2 \mathrm{e}^{-\mathrm{i}\omega_2 t})$$

$$\Omega_{1,2}^2 = \frac{\omega_1^2 + \omega_2^2}{2} + \kappa^2 \pm \sqrt{\left(\frac{\omega_1^2 - \omega_2^2}{2}\right)^2 + \kappa^4} = \frac{\omega_1^2 + \omega_2^2}{2} + \kappa^2 \pm \frac{w}{2}$$

$$\mathbf{x}(t) = \begin{pmatrix} 1 \\ \frac{-2\kappa^2}{\omega_1^2 - \omega_2^2 + w} \end{pmatrix} \left(A_1 \mathrm{e}^{\mathrm{i}\Omega_1 t} + A_2 \mathrm{e}^{-\mathrm{i}\Omega_1 t}\right) + \begin{pmatrix} \frac{-2\kappa^2}{\omega_2^2 - \omega_1^2 - w} \\ 1 \end{pmatrix} \left(B_1 \mathrm{e}^{\mathrm{i}\Omega_2 t} + B_2 \mathrm{e}^{-\mathrm{i}\Omega_2 t}\right)$$

Physical meaning of renormalization

Renormalization = formulation of the theory in terms of physically (more) relevant variables (in theories with interaction)

eigenvalues: $\omega_1, \omega_2 \longrightarrow \Omega_1, \Omega_2$

normal coordinates:

$$\mathbf{x}_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \mathbf{x}_2 = \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \longrightarrow \mathbf{y}_1 = N \begin{pmatrix} 1 \\ \frac{\omega_1^2 - \omega_2^2 + w}{2\kappa^2} \end{pmatrix} y_1, \; \mathbf{y}_2 = N \begin{pmatrix} \frac{\omega_2^2 - \omega_1^2 - w}{2\kappa^2} \\ 1 \end{pmatrix} y_2$$

$$N = \sqrt{1 + \left(\frac{\omega_1^2 - \omega_2^2 + w}{2\kappa^2}\right)^2}$$

$\Rightarrow$ "renormalized" Lagrangian

$$L = \frac{m}{2}(\dot{y}_1^2 - \Omega_1^2 y_1^2) + \frac{m}{2}(\dot{y}_2^2 - \Omega_2^2 y_2^2)$$

$\hookrightarrow$ renormalization already useful in classical physics with interaction

renormalization concerns parameters and coordinates of degrees of freedom

Scalar Field Theory

Lagrangian for scalar field $\phi_0(x)$ (ϕ^4 theory)

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{4!}\lambda_0\phi_0^4$$

bare parameters m_0, λ_0

free field theory: ($\lambda_0 = 0$)

$m_0 = m_{\text{phys}}$ physical mass

$\phi_0 = \phi_{\text{phys}}$ physical (properly normalized) field

interacting theory ($\lambda_0 \neq 0$):

$m_0 \neq m_{\text{phys}}$, $\lambda_0 \neq \lambda_{\text{phys}}$, $\phi_0 \neq \phi_{\text{phys}}$ $\Rightarrow$ renormalization necessary

problem: interaction cannot be switched off in nature

m_0 and λ_0 have no physical meaning, are not measurable

Concept of renormalization

Theory described by Lagrangian $\mathcal{L}(\phi_0, \partial_\mu \phi_0; m_0, \lambda_0)$

consider n physical observables: mass, cross sections, ...
with experimental values $E_1, \dots E_n$

calculate these observables: $E_1 = \sigma_1(m_0, \lambda_0), \dots, E_n = \sigma_n(m_0, \lambda_0)$

select two measurements E_1 and E_2 to determine m_0 and λ_0:

$$m_0 = m_0(E_1, E_2), \quad \lambda_0 = \lambda_0(E_1, E_2)$$

replace m_0 and λ_0 by observables E_1 and E_2 and normalize ϕ_0

$\Rightarrow$ renormalized Lagrangian

$$\mathcal{L}(\phi_0, \partial_\mu \phi_0; m_0(E_1, E_2), \lambda_0(E_1, E_2)) = \overline{\mathcal{L}}(\phi, \partial_\mu \phi; E_1, E_2)$$

same Lagrangian in terms of different parameters

need 2 (= # of free parameters) observables to fix the free parameters

remaining $n - 2$ observables can be used to test the theory

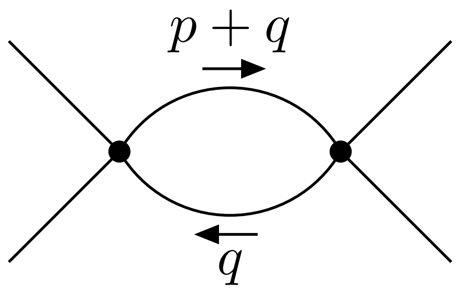
Renormalization schemes

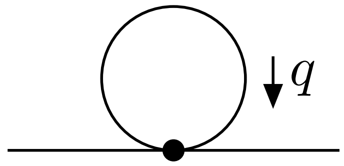
- Choice of physical quantities used to parametrize the renormalized Lagrangian (renormalization conditions) defines a renormalization scheme
  suitable choice: well-defined simple quantities that are measurable and calculable with high precision: masses, . . .
- can use renormalized parameters m_{ren} and λ_{ren} as “pseudo-observables”
  couplings are no direct observables but can be determined from (a set of) observables
- in exact theory all schemes are equivalent
  calculations are done in approximations (perturbation theory)
  $\Rightarrow$ renormalization scheme dependence of predictions in approximations

Ultraviolet divergences

Observation: loop integrals involve divergences

Examples: boson-boson scattering and self-energy in ϕ^4 theory





ultraviolet (UV) divergences for $q \to \infty$ (large momenta)

$$\int \mathrm{d}^4 q \, \frac{1}{(q^2 - m^2 + \mathrm{i}\epsilon)[(q+p)^2 - m^2 + \mathrm{i}\epsilon]} \sim \int \frac{\mathrm{d}q}{q} \quad \text{for } q \to \infty$$

$\longrightarrow$ logarithmic divergence

$$\int \mathrm{d}^4 q \, \frac{1}{(q^2 - m^2 + \mathrm{i}\epsilon)} \sim \int \mathrm{d}q \, q \quad \text{for } q \to \infty$$

$\longrightarrow$ quadratic divergence

Regularization

Manipulation of divergent integrals needs regularization:
modification of theory by introducing free parameter δ such that

- original theory is obtained as limiting case $\delta \to \delta_0$ (typically $\delta_0 = 0$ or ∞)
- integrals (and thus the theory) become finite, i.e. well defined for $\delta \neq \delta_0$

consequence: $E_i = \sigma_i(m_0, \lambda_0, \delta)$

$\hookrightarrow m_0 = m_0(E_1, E_2, \delta), \quad \lambda_0 = \lambda_0(E_1, E_2, \delta)$

bare parameters depend on cut-off and are UV singular for $\delta \to \delta_0$

$\hookrightarrow$ bare parameters have no physical meaning

$\Rightarrow$ renormalized Lagrangian $\quad \overline{\mathcal{L}}(\phi, \partial_\mu \phi, E_1, E_2, \delta) \quad$ depends on cut-off

observables must have finite limit for $\delta \to \delta_0$ as functions of E_1, E_2:
$\sigma_i(E_1, E_2, \delta) \underset{\delta \to \delta_0}{\longrightarrow} \sigma_i(E_1, E_2)$ (independent of regularization scheme)

relations between physical quantities should be finite and indep. of cut-off
(physical results should be independent of the regularization scheme)

if true: theory is renormalizable

Regularization schemes

Cut-off regularization

require $q_0^2 + \mathbf{q}^2 < \Lambda^2$ in momentum space

$\hookrightarrow$ UV divergences appear as $\log \Lambda^2$, Λ^2, ...

breaks Lorentz invariance and gauge invariance

lattice regularization [Wilson 1974]

discretize space–time by a lattice with lattice constant a

$\hookrightarrow$ UV divergences appear as $1/a$

breaks Lorentz invariance

dimensional regularization ['t Hooft, Veltman 1972]

switch to $D \neq 4$ space–time dimensions (μ arbitrary reference mass)

$\int \mathrm{d}^4 q\, f(q) \to \mu^{4-D} \int \mathrm{d}^D q\, f(q), \quad D = 4 - 2\delta$

D-dimensional momenta, metric $g^{\mu\nu}$, Dirac algebra

and analytic continuation in D

$\hookrightarrow$ UV divergences appear as $\delta^{-1} - \gamma_\mathrm{E} + \log(4\pi\mu^2)$ for $\delta \to 0$

respects Lorentz and gauge invariance and is simple

criteria for choice: keep as many relevant symmetries as possible
(Lorentz invariance, gauge invariance, supersymmetry, ...)

Example: self-energy integral in ϕ^4 theory

2-point function in dimensional regularization

$$B_0(p^2,m,m) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int \mathrm{d}^D q \, \frac{1}{(q^2-m^2+\mathrm{i}\epsilon)[(q+p)^2-m^2+\mathrm{i}\epsilon]}$$

Feynman parametrization

$$= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int \mathrm{d}^D q \int_0^1 \mathrm{d}x \Big[(q^2-m^2+\mathrm{i}\epsilon)(1-x) + [(q+p)^2-m^2+\mathrm{i}\epsilon]x\Big]^{-2}$$

shift of integration momentum $q \to q - px$

$$= \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2}\int_0^1 \mathrm{d}x \int \mathrm{d}^D q \left[q^2 - x^2p^2 + xp^2 - m^2 + \mathrm{i}\epsilon\right]^{-2}$$

Wick rotation and momentum integration

$$= (4\pi\mu^2)^{\delta}\Gamma(\delta)\int_0^1 \mathrm{d}x \, [x^2p^2 - xp^2 + m^2 - \mathrm{i}\epsilon]^{-\delta}$$

Example: self-energy integral in ϕ^4 theory (cont.)

Expansion in $\delta = (4 - D)/2$

$\delta \ll 1: \quad \Gamma(\delta) = \frac{1}{\delta} - \gamma_{\mathrm{E}} + \mathcal{O}(\delta), \qquad a^\delta = \mathrm{e}^{\delta \log a} = 1 + \delta \log a + \mathcal{O}(\delta^2)$

$$B_0(p^2, m, m) = \underbrace{\frac{1}{\delta} - \gamma_{\mathrm{E}} + \log(4\pi)}_{\Delta} - \int_0^1 \mathrm{d}x \log \frac{x^2 p^2 - x p^2 + m^2 - \mathrm{i}\epsilon}{\mu^2} + \mathcal{O}(\delta)$$

$$= \Delta + 2 - \log \frac{m^2}{\mu^2} - \frac{m^2}{p^2}\left(\frac{1}{r} - r\right) \log r + \mathcal{O}(\delta)$$

$$r = \frac{1}{2m^2}\left[2m^2 - p^2 - \mathrm{i}\epsilon \pm \sqrt{(2m^2 - p^2 - \mathrm{i}\epsilon)^2 - 4m^4}\right]$$

UV-divergence in $1/\delta$

MS renormalization: subtract term proportional to $1/\delta$

$\overline{\text{MS}}$ renormalization: subtract term proportional to Δ

Vertex functions

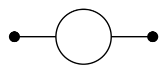
One-particle irreducible Feynman diagram: diagram that cannot be split into two parts by cutting *one* internal line

vertex function: sum of all one-particle irreducible diagrams with fixed external legs

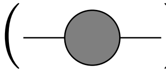
vertex functions are basic building blocks of all Green functions and S-matrix elements

propagator = inverse 2-point function

$$G^{\phi\phi}(p) \equiv$$

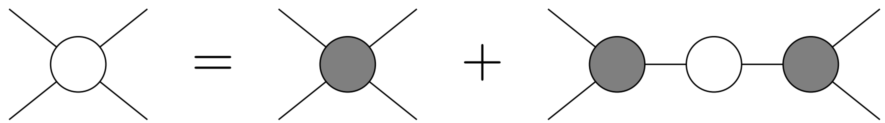


$$= -\left[\Gamma^{\phi\phi}(p)\right]^{-1} \equiv -\Big($$



$$\Big)^{-1}$$

truncated 4-point function (external 2-point functions omitted)



$$G^{\phi\phi\phi\phi}_{\mathrm{trunc}} = \Gamma^{\phi\phi\phi\phi} + \Gamma^{\phi\phi\phi} G^{\phi\phi} \Gamma^{\phi\phi\phi} + \text{two permutations}$$

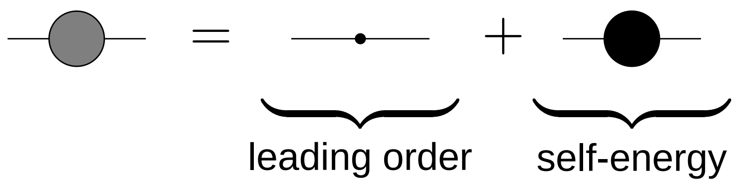
vertex functions contain all loops $\Rightarrow$ all UV divergences

$\hookrightarrow$ for renormalizability the study of vertex functions is sufficient

Structure of one-loop divergences in ϕ^4 theory

2-point function (inverse propagator)

$\Gamma^{\phi_0\phi_0}(p) = \mathrm{i}(p^2 - m_0^2) + \mathrm{i}\Sigma(p^2)$, Σ = self-energy

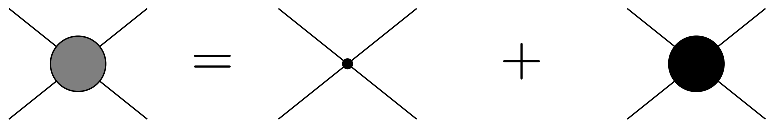


$$\Sigma(p^2) = (C_1\,p^2 + C_2)\,\Delta + \Sigma_{\mathrm{finite}}(p^2)$$ = UV divergent

$\hookrightarrow$ higher-order corrections change mass (zero of $\Gamma^{\phi\phi}$) and normalization

4-point function

$$\Gamma^{\phi_0\phi_0\phi_0\phi_0}(p_1,p_2,p_3) = -\mathrm{i}\lambda_0 + \mathrm{i}\Lambda(p_1,p_2,p_3)$$



momentum-dependent one-loop correction:

$$\Lambda(p_1,p_2,p_3) = C_3\Delta + \Lambda_{\mathrm{finite}}(p_1,p_2,p_3)$$ = UV divergent

$\hookrightarrow$ higher-order corrections change coupling strength

vertex functions with more external legs are finite in ϕ^4 theory

One-loop renormalization of ϕ^4 theory

Introduce renormalized parameters and fields that obey certain conditions

2-point function

- mass renormalization: $m_0^2 = m^2 + \delta m^2$

  $m^2 \stackrel{!}{=}$ location of propagator pole $=$ "physical mass"

  $$\Gamma^{\phi\phi}(p)\big|_{p^2=m^2} \stackrel{!}{=} 0 \to \ \delta m^2 = \Sigma(m^2)$$

- field renormalization: rescale field $\phi_0 = \sqrt{Z_\phi}\phi$, $\quad \Gamma^{\phi\phi} = Z_\phi \Gamma^{\phi_0\phi_0}$

  fix $Z_\phi = 1 + \delta Z_\phi$ such that residue of propagator $G^{\phi\phi} = -(\Gamma^{\phi\phi})^{-1}$ at $p^2 = m^2$ equals 1

  $$\left.\frac{\mathrm{d}\Gamma^{\phi\phi}(p)}{\mathrm{d}p^2}\right|_{p^2=m^2} \stackrel{!}{=} 1 \to \ \delta Z_\phi = -\Sigma'(m^2)$$

$\Rightarrow$ renormalized 2-point function $\Gamma^{\phi\phi}$ is UV finite:

$$\begin{aligned}
\Gamma^{\phi\phi}(p^2) &= \mathrm{i}Z_\phi[p^2 - m^2 - \delta m^2 + \Sigma(p^2)] \\
&= \mathrm{i}[(1+\delta Z_\phi)(p^2 - m^2) - \delta m^2 + \Sigma(p^2)] \\
&= \mathrm{i}[p^2 - m^2 + \Sigma(p^2) - \Sigma(m^2) - (p^2 - m^2)\Sigma'(m^2)] \\
&= \mathrm{i}[p^2 - m^2 + \Sigma_{\mathrm{finite}}(p^2) - \Sigma_{\mathrm{finite}}(m^2) - (p^2 - m^2)\Sigma'_{\mathrm{finite}}(m^2)]
\end{aligned}$$

One-loop renormalization of ϕ^4 theory (cont.)

4-point function:

- coupling-constant renormalization: $\lambda_0 \;=\; \lambda + \delta\lambda$

  fix $\delta\lambda$ such that λ assumes a measured value for special kinematics $p_i^{\rm exp}$ (renormalization point)

  note: $\Gamma^{\phi\phi\phi\phi} \;=\; Z_\phi^2 \Gamma^{\phi_0\phi_0\phi_0\phi_0}$

  $\hookrightarrow\;\; \delta\lambda = -2\delta Z_\phi \lambda + \Lambda(p_1^{\rm exp}, p_2^{\rm exp}, p_3^{\rm exp})$

$\Rightarrow$ renormalized 4-point function is UV finite:

$$\begin{aligned}\Gamma^{\phi\phi\phi\phi}(p_1,p_2,p_3) &= {\rm i} Z_\phi^2[-\lambda - \delta\lambda + \Lambda(p_1,p_2,p_3)]\\ &= {\rm i}[-\lambda(1+2\delta Z_\phi) - \delta\lambda + \Lambda(p_1,p_2,p_3)]\\ &= {\rm i}[-\lambda + \Lambda(p_1,p_2,p_3) - \Lambda(p_1^{\rm exp},p_2^{\rm exp},p_3^{\rm exp})]\\ &= {\rm i}[-\lambda + \Lambda_{\rm finite}(p_1,p_2,p_3) - \Lambda_{\rm finite}(p_1^{\rm exp},p_2^{\rm exp},p_3^{\rm exp})]\end{aligned}$$

$\hookrightarrow$ all divergences of ϕ^4 theory can be absorbed by renormalization of parameters m, λ and the field ϕ

field renormalization needed for finite vertex functions

automatic in S-matrix elements because of necessary normalization

Structure of divergences in general

High-energy behaviour of integrands of Feynman integrals determined by loop integration, internal propagators, momentum-dependent vertices

$$\int \mathrm{d}^4 q \cdots \frac{N(q,\ldots,p,\ldots)}{(q^2-m^2+\mathrm{i}\epsilon)[(q+p)^2-m^2+\mathrm{i}\epsilon]\cdots}$$

consider one-particle irreducible Feynman diagram G with E_ϕ external scalars, E_ψ external fermions and E_V external vector particles

power counting and combinatorics yields superficial degree of divergence

(all integration momenta tend to ∞ simultaneously)

$$\omega(G) = 4 - E_\phi - \tfrac{3}{2}E_\psi - E_V - \sum_v [g_v]$$

$\sum_v$: sum over all vertices of G

$[g_v]$: mass dimension of coupling constant at vertex v

assumes that scalar and vector-boson propagators behave as $1/k^2$ and fermion propagators as $1/k$ for large k

Weinberg 1960: all UV divergences are controlled by the superficial degree of divergence
(sub-divergences are treated iteratively order by order)

Renormalizable theories

$[g_v] \geq 0$ for all couplings (renormalizable couplings)

$\hookrightarrow$ only finite number of primitively divergent vertex functions

$$\omega(\Gamma) = 4 - E_\phi - \tfrac{3}{2}E_\psi - E_V$$

$\hookrightarrow$ all UV divergences can be absorbed by renormalization of finite number of parameters

examples

- ϕ^4 theory: $\omega(\Gamma) = 4 - E_\phi$ (finite owing to symmetry $\phi \to -\phi$)

- QED: $\omega(\Gamma) = 4 - \frac{3}{2}E_\psi - E_V$ (finite owing to symmetries)

  primitively divergent vertex functions

Non-renormalizable theories

$[g_v] < 0$ for at least one coupling

$\hookrightarrow$ every vertex function gets divergent from diagrams with sufficiently many couplings of negative mass dimension

$\hookrightarrow$ number of primitively divergent vertex functions grows order by order

$\hookrightarrow$ number of free parameters grows order by order

$\hookrightarrow$ much less predictive power

example

- Fermi model: $[G_\mu] = -2, \qquad \omega(G) = 4 - \frac{3}{2}E_\psi + 2V$

  one-loop order: $2V = E_\psi \quad \Rightarrow \quad \omega(\Gamma) = 4 - \frac{1}{2}E_\psi$

  primitively divergent vertex functions in one-loop order $E_\psi = 2, 4, 6, 8$

  need many more free parameters already at one-loop order

Structure of counterterms in renormalizable theories

Example:

$$I(p,m) = \int \mathrm{d}^4 q \, \frac{1}{(q^2 - m^2)[(q+p)^2 - m^2]} \qquad \text{log. divergent}$$

$$\frac{\partial}{\partial p_\mu} I(p,m) = \int \mathrm{d}^4 q \, \frac{2(q+p)^\mu}{(q^2 - m^2)[(q+p)^2 - m^2]^2} \qquad \text{finite}$$

general: differentiation with respect to external momentum reduces $\omega(G)$ by 1

$\hookrightarrow$ $\omega(G) + 1$ differentiations with respect to external momenta render loop integral finite (if subdivergences are renormalized)

$\hookrightarrow$ UV divergence is polynomial in external momenta of degree $\omega(G)$

$\hookrightarrow$ counterterms for Γ are polynomial in external momenta of degree $\omega(\Gamma)$

structure of counterterm polynomial

$$\delta\mathcal{L} = \sum_k c_k \partial^{\gamma_k} \phi^{E_\phi} \psi^{E_\psi} V^{E_V} \qquad \text{with} \qquad 0 \le \gamma_k \le \omega(\Gamma)$$

$$[c_k] = 4 - E_\phi - \tfrac{3}{2} E_\psi - E_V - \gamma_k = \omega(\Gamma) - \gamma_k \ge 0$$

$\hookrightarrow$ counterterms are renormalizable

(c_k is coupling with positive mass dimension)

Structure of counterterms in renormalizable theories (cont.)

Consider Lagrangian $\mathcal{L}$ with given set of fields and interactions:

- $\mathcal{L}$ contains all possible renormalizable interactions with independent coupling parameters
  - $\hookrightarrow$ all divergences can be absorbed by redefinition of parameters and fields of the Lagrangian
  - $\Rightarrow$ multiplicative renormalization possible
- $\mathcal{L}$ does not contain all possible renormalizable interactions
  - $\hookrightarrow$ must introduce additional (finite) set of parameters to absorb all divergences
  - $\hookrightarrow$ all possible renormalizable interactions appear in higher orders
- $\mathcal{L}$ has a global symmetry
  - $\hookrightarrow$ counterterms have symmetry of Lagrangian (proof for *local* gauge symmetry non-trivial)

Multiplicative renormalization

Example: ϕ^4 theory

$$\mathcal{L}(\phi_0, \partial\phi_0; m_0, \lambda_0) = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{4!}\lambda_0\phi_0^4$$

multiplicative renormalization transformation

$$m_0(m,\lambda,\delta) = Z_m(m,\lambda,\delta)m, \qquad \lambda_0(m,\lambda,\delta) = Z_\lambda(m,\lambda,\delta)\lambda$$
$$\phi_0 = Z_\phi^{1/2}(m,\lambda,\delta)\phi$$

renormalized Lagrangian

$$\mathcal{L} = \mathcal{L}(\phi_0, \partial\phi_0; m_0, \lambda_0) = \mathcal{L}(Z_\phi^{1/2}\phi, Z_\phi^{1/2}\partial\phi_0; Z_m m, Z_\lambda\lambda)$$

perturbation theory: $Z_i = 1 + \delta Z_i$

$$\mathcal{L} = \mathcal{L}(\phi, \partial\phi; m, \lambda) \quad + \quad \mathcal{L}_{\mathrm{ct}}(\phi, \partial\phi; m, \lambda; \delta Z_\phi, \delta Z_m, \delta Z_\lambda)$$

same form as $\mathcal{L}$ but renormalized parameters	counterterm Lagrangian $\rightarrow$ counterterm diagrams

Feynman rules contain renormalized parameters and counterterms

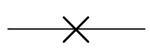
Counterterms of ϕ^4 theory

Lagrangian

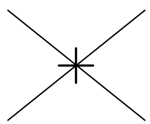
$$
\begin{aligned}
\mathcal{L} &= \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 - \frac{1}{2}m_0^2\phi_0^2 - \frac{1}{4!}\lambda_0\phi_0^4 \\
&= \frac{1}{2}Z_\phi\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}Z_\phi Z_m m^2\phi^2 - \frac{1}{4!}Z_\phi^2 Z_\lambda\lambda\phi^4 \\
&= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \\
&\quad - \frac{1}{2}\delta Z_m m^2\phi^2 + \frac{1}{2}\delta Z_\phi[\partial_\mu\phi\partial^\mu\phi - m^2\phi^2] - \frac{1}{4!}(2\delta Z_\phi + \delta Z_\lambda)\lambda\phi^4 \\
&\quad + \mathcal{O}((\delta Z)^2)
\end{aligned}
$$

= renormalized Lagrangian

Feynman rules for counterterms



$$\mathrm{i}[\delta Z_\phi(p^2 - m^2) - \delta Z_m m^2] + \mathcal{O}((\delta Z)^2)$$

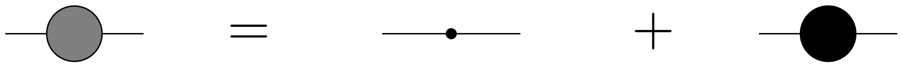


$$-\mathrm{i}(\delta Z_\lambda + 2\delta Z_\phi)\lambda + \mathcal{O}((\delta Z)^2)$$

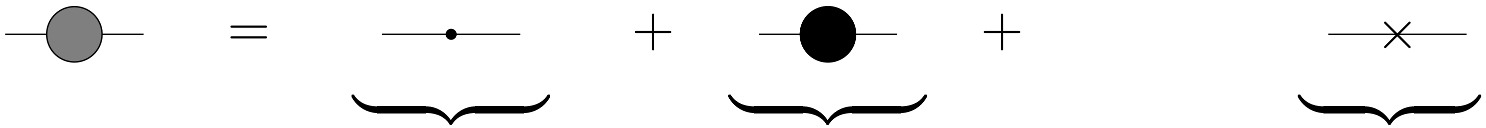
Divergent vertex functions in ϕ^4 theory at one-loop level

2-point function

$$\Gamma^{\phi_0\phi_0}(p) = \mathrm{i}(p^2 - m_0^2) + \mathrm{i}\Sigma(p^2)$$



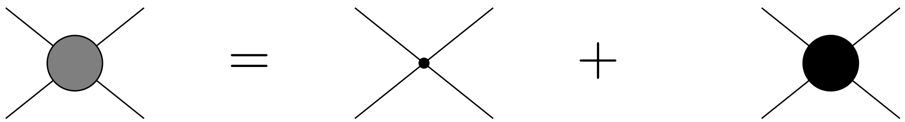
$$\Rightarrow \quad \Gamma^{\phi\phi}(p) = \mathrm{i}(p^2 - m^2) + \mathrm{i}\Sigma(p^2) + \mathrm{i}[\delta Z_\phi(p^2 - m^2) - \delta Z_m m^2]$$



leading order — self-energy — counterterm

4-point function

$$\Gamma^{\phi_0\phi_0\phi_0\phi_0}(p_1, p_2, p_3) = -\mathrm{i}\lambda_0 + \mathrm{i}\Lambda\,(p_1, p_2, p_3)$$



$$\Rightarrow \quad \Gamma^{\phi\phi\phi\phi}(p_1, p_2, p_3) = -\mathrm{i}\lambda + \mathrm{i}\Lambda\,(p_1, p_2, p_3) - \mathrm{i}(\delta Z_\lambda + 2\delta Z_\phi)\lambda$$

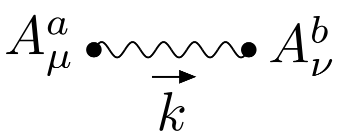
Quantization of gauge theories

Gauge fields contain unphys. degrees of freedom that must not be quantized.
$\Rightarrow$ quantization of gauge theories requires to fix a gauge

$\hookrightarrow$ effectively amounts to adding extra terms to Lagrangian

R_ξ gauge-fixing Lagrangian

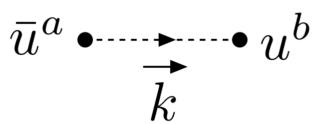
$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{\xi} F^a F^a, \qquad F^a = \partial^\mu A^a_\mu$$

$\hookrightarrow$ gauge-boson propagator:  $A^a_\mu \; \vec{k} \; A^b_\nu$

$$-\mathrm{i}\delta^{ab} \frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}(1-\xi)}{k^2 + \mathrm{i}\epsilon}$$

contributions of unphysical degrees of freedom of gauge fields are compensated by Faddeev–Popov ghost fields

$$\mathcal{L}_{\mathrm{FP}} = -\bar{u}^a(x)\partial^\mu[\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu(x)] u^b(x)$$

$\hookrightarrow$ Faddeev–Popov ghost propagator:  $\bar{u}^a \; \vec{k} \; u^b$

$$= \delta^{ab} \frac{\mathrm{i}}{k^2}$$

Lagrangian of quantized gauge theory $\mathcal{L} = \mathcal{L}_{\mathrm{class}} + \mathcal{L}_{\mathrm{gf}} + \mathcal{L}_{\mathrm{FP}}$ is invariant under Becchi–Rouet–Stora (BRS) symmetry [Becchi, Rouet, Stora 1974]

Renormalizability of gauge theories

- Gauge theories are renormalizable according to power counting
  (in renormalizable gauges: gauge-boson propagators $\sim 1/k^2$)
  $\omega(\Gamma) = 4 - E_\phi - \frac{3}{2}E_\psi - E_V - E_u$
- Gauge theories involve unphysical degrees of freedom
  (scalar polarizations of gauge fields, Faddeev–Popov ghosts, would-be Goldstone bosons)
  $\hookrightarrow$ must proof that unphysical states do not contribute to the physical S matrix: proof unitarity of physical S matrix
  needs definition of physical states
  (states that are invariant under BRS transformations)
- gauge-fixed Lagrangian depends on gauge choice and gauge parameters
  $\hookrightarrow$ proof that physical S matrix is gauge independent
- proofs rely on BRS invariance
  $\hookrightarrow$ proof that theory can be renormalized such that it is invariant under (renormalized) BRS invariance

Multiplicative renormalization of a gauge theory

All UV divergences can be absorbed by the renormalization transformations (gauge theory with fermion multiplet of mass m)

$$\begin{aligned} A^a_{0,\mu} &= Z_A^{1/2} A^a_\mu, & \xi_0 &= Z_A \xi, & g_0 &= Z_g g \\ u^a_0 &= Z_u^{1/2} u^a, & \bar{u}^a_0 &= Z_u^{1/2} \bar{u}^a \\ \psi_0 &= Z_\psi^{1/2} \psi, & m_0 &= Z_m m \end{aligned}$$

- gauge-fixing parameter is renormalized by the same constant as the gauge field
  $\hookrightarrow$ gauge-fixing term $-(\partial^\mu A^a_\mu)^2/(2\xi)$ is not renormalized
- field renormalization for Faddeev–Popov ghost fields needed to render all Green functions finite (not needed for S-matrix elements)
- gauge coupling in different vertices is renormalized in the same way (consequence of gauge/BRS invariance)
- additional non-gauge coupling constants, like 4-scalar coupling λ need independent renormalization

Renormalized Lagrangian is invariant under (renormalized) BRS invariance.

Anomalies

Anomaly = breakdown of a classical symmetry upon quantization

anomalies in gauge theories destroy BRS symmetry and thus renormalizability

$\hookrightarrow$ renormalizable gauge theories must be free of anomalies

condition for cancellation of (axial) anomaly (γ_5 anomaly)

$$A^{abc} = \mathrm{Tr}\Big(T^a_\mathrm{L}\left\{T^b_\mathrm{L}, T^c_\mathrm{L}\right\}\Big) - \mathrm{Tr}\Big(T^a_\mathrm{R}\left\{T^b_\mathrm{R}, T^c_\mathrm{R}\right\}\Big) = 0$$

$T^a_\mathrm{L}, T^a_\mathrm{R}$ representation matrices for left-handed, right-handed fermion fields

vector theory (QED,QCD): $T^a_\mathrm{L} = T^a_\mathrm{R} \Rightarrow A^{abc} = 0$

Standard Model: anomaly cancellation requires $\sum_f Q_f = 0$
single family:

$$(0-1) + N_\mathrm{c}\left(\frac{2}{3} - \frac{1}{3}\right) = -1 + \frac{N_\mathrm{c}}{3} = 0$$

$\Rightarrow$ three quark colours ($N_\mathrm{c} = 3$) are needed

Renormalization of the electroweak Standard Model

Bare input parameters: $e_0, M_{\mathrm{W},0}, M_{\mathrm{Z},0}, M_{\mathrm{H},0}, m_{f,0}, V_{ij,0}$

renormalization transformation:

- parameter renormalization:

$$
\begin{aligned}
e_0 &= Z_e e \\
M_{\mathrm{W},0}^2 &= M_{\mathrm{W}}^2 + \delta M_{\mathrm{W}}^2, \quad M_{\mathrm{Z},0}^2 = M_{\mathrm{Z}}^2 + \delta M_{\mathrm{Z}}^2, \quad M_{\mathrm{H},0}^2 = M_{\mathrm{H}}^2 + \delta M_{\mathrm{H}}^2 \\
m_{f,0} &= m_f + \delta m_f, \quad V_{ij,0} = V_{ij} + \delta V_{ij}, \quad \text{(both } V_{ij,0}, V_{ij} \text{ unitary)}
\end{aligned}
$$

Note: renormalization of $c_{\mathrm{w}}, s_{\mathrm{w}}$ fixed by on-shell condition $c_{\mathrm{w}} = M_{\mathrm{W}}/M_{\mathrm{Z}}$
(s_{w} is *not* a free parameter if M_{W}, M_{Z} are used as input parameters)

- field renormalization: (physical fields)

$$
W_0^{\pm} = \sqrt{Z_W}\, W^{\pm}, \quad \begin{pmatrix} Z_0 \\ A_0 \end{pmatrix} = \begin{pmatrix} \sqrt{Z_{ZZ}} & \sqrt{Z_{ZA}} \\ \sqrt{Z_{AZ}} & \sqrt{Z_{AA}} \end{pmatrix} \begin{pmatrix} Z \\ A \end{pmatrix}, \quad H_0 = \sqrt{Z_H}\, H
$$

$$
\psi_{f,0}^{\mathrm{L}} = \sqrt{Z_{ff'}^{\mathrm{L}}}\, \psi_{f'}^{\mathrm{L}}, \qquad \psi_{f,0}^{\mathrm{R}} = \sqrt{Z_{ff'}^{\mathrm{R}}}\, \psi_{f'}^{\mathrm{R}}
$$

Note: matrix renorm. necessary to account for loop-induced mixing

Renormalization conditions for Electroweak Standard Model

- Mass renormalization:

  on-shell definition: mass^2 is location of pole in propagator

  $\hookrightarrow \ \delta M_\mathrm{W}^2 \ = \ \mathrm{Re}\{\Sigma_\mathrm{T}^W(M_\mathrm{W}^2)\},$ similar expressions for $\delta M_\mathrm{Z}^2, \delta M_\mathrm{H}^2, \delta m_f$

- field renormalization: (bosons and leptons)

  $Z_i = 1 + \delta Z_i, \quad Z_{ii} = 1 + \delta Z_{ii}, \quad Z_{ij} = \delta Z_{ij}, \quad i \neq j$

  ◇ residues of propagators (diagonal, transverse parts) normalized to 1

  $\hookrightarrow \ \delta Z_W \ = \ -\mathrm{Re}\{\Sigma_\mathrm{T}^W{}'(M_\mathrm{W}^2)\},$

  similar expressions for $\delta Z_{AA}, \delta Z_{ZZ}, \delta Z_H, \delta Z_{ff}^{\mathrm{L/R}}$

  ◇ suppression of mixing propagators on particle poles

  physical on-shell particles do not mix

  i.e. $\Gamma_\mathrm{T}^{AZ}(0) = 0 = \Gamma_\mathrm{T}^{AZ}(M_\mathrm{Z}^2)$

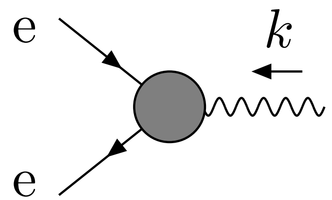
  $\hookrightarrow$ fixes non-diagonal constants:

  $\delta Z_{AZ} = -2\,\mathrm{Re}\{\Sigma_\mathrm{T}^{AZ}(M_\mathrm{Z}^2)\}/M_\mathrm{Z}^2, \quad \delta Z_{ZA} = 2\{\Sigma_\mathrm{T}^{AZ}(0)\}/M_\mathrm{Z}^2$

  similar expressions for $\delta Z_{ff'}^{\mathrm{L/R}} \quad (f \neq f')$

Renormalization conditions for Electroweak Standard Model (cont.)

- Charge renormalization: define e in Thomson limit



$A_\mu \xrightarrow{k\to 0} \mathrm{i}e\gamma_\mu$ for on-shell electrons

$\hookrightarrow e =$ elementary charge of classical electrodynamics

fine-structure constant $\alpha(0) = \frac{e^2}{4\pi} = 1/137.03599976$

gauge invariance relates δZ_e to photon wave-function renormalization:

$$\delta Z_e = -\frac{1}{2}\delta Z_{AA} - \frac{s_\mathrm{w}}{2c_\mathrm{w}}\delta Z_{ZA} \qquad \text{(at one loop)}$$

- CKM-matrix renormalization $\rightarrow$ fixes δV_{ij}
  non-trivial, but phenomenologically not yet relevant

general result in SM: All renormalization constants can be obtained from self-energies.

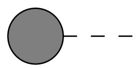
Renormalization of vev/tadpole

Renormalization of spontaneously broken theories requires in addition renormalization of vacuum expectation value (vev) of scalar field

- elimination of divergences $\Leftarrow$ renormalize v as ϕ:

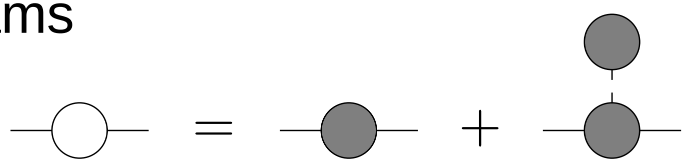
  $v_0 = Z_\phi v$ e.g. $Z_\phi = Z_H$

- spontaneous symmetry breaking gives rise to tadpole diagrams

  $\Gamma^H(0) =$ 

tadpole diagrams contribute to all Green functions and matrix elements

$\hookrightarrow$ many more diagrams

e.g. 2-point function 

convenient renormalization condition for vev:

vanishing of renormalized tadpole

$\hookrightarrow$ introduce counterterm $\delta t H(x)$ in $\mathcal{L}$

require $\mathrm{i}\delta t + \Gamma^H(0) = 0 \quad \Leftrightarrow$ renormalized v is true vev

connection between tadpole and vev: $\delta t = v_0 \left(\mu_0^2 - \frac{1}{4}\lambda_0 v_0^2 \right)$

Gauge fixing in Electroweak Standard Model

Gauge-fixing Lagrangian:

$$\mathcal{L}_{\mathrm{gf}} = -\frac{1}{\xi_{\mathrm{W}}} F^+ F^- - \frac{1}{2\xi_{\mathrm{Z}}}(F^{\mathrm{Z}})^2 - \frac{1}{2\xi_\gamma}(F^\gamma)^2$$

with the gauge-fixing functionals:

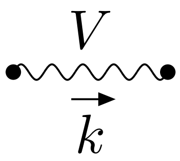
$$F^\pm = \partial^\mu W^\pm_\mu \mp \mathrm{i}\xi'_{\mathrm{W}} M_{\mathrm{W}} \phi^\pm, \qquad F^{\mathrm{Z}} = \partial^\mu Z_\mu - \xi'_{\mathrm{Z}} M_{\mathrm{Z}} \chi, \qquad F^\gamma = \partial^\mu A_\mu$$

(ξ_V, ξ'_V = gauge-fixing parameters)

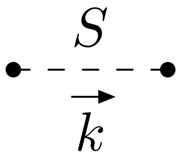
features of gauge fixing:

- 't Hooft gauge: $\xi_W = \xi'_W$, $\xi_Z = \xi'_Z$
  elimination of mixing terms $(W^\pm_\mu \partial^\mu \phi^\mp)$, $(Z_\mu \partial^\mu \chi)$ in Lagrangian
  $\hookrightarrow$ decoupling of gauge and would-be Goldstone fields
  (no mixing propagators)
- ghost fields couple to gauge fields and scalar fields

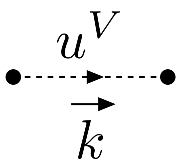
Boson propagators in ’t Hooft gauge



$$= -\mathrm{i}\left[\frac{g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}}{k^2 - M_V^2} + \frac{k_\mu k_\nu}{k^2}\frac{\xi_V}{k^2 - \xi_V M_V^2}\right], \qquad V = W, Z, \gamma$$



$$= \frac{\mathrm{i}}{k^2 - \xi_V M_V^2}, \qquad S = \phi, \chi$$



$$= \frac{\mathrm{i}}{k^2 - \xi_V M_V^2}, \qquad V = W^\pm, Z, \gamma$$

important special cases:

- $\xi_V = 1$: ‘t Hooft–Feynman gauge
  $\hookrightarrow$ convenient gauge-boson propagators $\dfrac{-\mathrm{i}g_{\mu\nu}}{k^2 - M_V^2}$
- $\xi_\mathrm{W}, \xi_\mathrm{Z} \to \infty$: “unitary gauge”
  $\hookrightarrow$ elimination of would-be Goldstone bosons and Faddeev–Popov ghosts

Renormalization of the unphysical sector

In order to render all Green functions finite also the unphysical fields have to be renormalized:

$$\chi_0 = \sqrt{Z_\chi}\,\chi, \qquad \phi_0^\pm = \sqrt{Z_\phi}\,\phi^\pm$$

$$u_0^\pm = \sqrt{\tilde{Z}_u}\,u^\pm, \qquad \begin{pmatrix} u_0^Z \\ u_0^A \end{pmatrix} = \begin{pmatrix} \sqrt{\tilde{Z}_{ZZ}} & \sqrt{\tilde{Z}_{ZA}} \\ \sqrt{\tilde{Z}_{AZ}} & \sqrt{\tilde{Z}_{AA}} \end{pmatrix} \begin{pmatrix} u^Z \\ u^A \end{pmatrix}$$

$$\xi_{0,W}^{(\prime)} = Z_{\xi_W^{(\prime)}}\,\xi_W^{(\prime)}, \qquad \xi_{0,Z}^{(\prime)} = Z_{\xi_Z^{(\prime)}}\,\xi_Z^{(\prime)}, \qquad \xi_{0,A} = Z_{\xi_A}\,\xi_A$$

- Renormalization of gauge parameters is such that the gauge-fixing terms are not renormalized.
- 't Hooft gauge needs different renormalization of ξ_V and ξ_V';
  also 't Hooft–Feynman gauge needs counterterms for $\delta\xi_V$ and $\delta\xi_V'$.
- Field renormalization of unphysical fields is performed similar to the one for the physical fields:
  residues of (diagonal) propagators normalized to one
  suppression of mixing propagators on poles

Summary on renormalization

- Renormalization is a reformulation of the theory in terms of more appropriate (physical) parameters and fields.
- UV-divergences of renormalizable theories can be absorbed in redefinition of parameters and fields.
- Renormalizable theories require couplings of non-negative mass dimension and gauge symmetry for vector particles.
- Renormalizable theories allow for accurate predictions using perturbation theory.
- QED, QCD, and the electroweak Standard Model are renormalizable.

Literature

- Textbooks:
  - Böhm/Denner/Joos: “Gauge Theories of the Strong and Electroweak Interaction”
  - Collins: “Renormalization”
  - Itzykson/Zuber: “Quantum Field Theory”
  - Peskin/Schroeder: “An Introduction to Quantum Field Theory”
  - Weinberg: “The Quantum Theory of Fields, Vol. 1: Foundations”; “The Quantum Theory of Fields, Vol. 2: Modern Applications”
- renormalization of the electroweak SM (incomplete and personally biased):

  - K. I. Aoki, Z. Hioki, M. Konuma, R. Kawabe and T. Muta, Prog. Theor. Phys. Suppl. 73 (1982) 1
  - M. Böhm, W. Hollik and H. Spiesberger, Fortsch. Phys. 34 (1986) 687
  - W. F. Hollik, Fortsch. Phys. 38 (1990) 165
  - A. Denner, Fortsch. Phys. 41 (1993) 307
  - A. Denner, S. Dittmaier and G. Weiglein, Nucl. Phys. B 440 (1995) 95